Static System Analysis based on Singular Value Decomposition of Quaternion Matrix

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Abstract: This article gives a constructive proof of singular value decomposition of a quaternion matrix by using the complex representation of quaternion matrices and properties of the eigenvectors. Therefore, a practical algorithm for singular value decomposition of quaternion matrices is obtained. At the same time, the algorithm of singular value decomposition is applied to the analysis of static system to obtain the algebraic equivalent form between voltage and current in the static system model, thus obtaining the most reliable numerical expression.

1. Introduction

Singular value decomposition of matrices is an important tool for the study of matrix theory and matrix calculations. With the development of modern science and technology, it has direct and important applications in control theory, system identification, signal processing, optimization problems, eigvalue problems, least squares problems and statistics. It is the same case in singular value decomposition of quaternion matrices. Its application in computer graphics and quantum mechanics is particularly important [3]. For example, the singular value decomposition of quaternion is applied to the analysis of static systems [2] [6].

Assume R Is a real number field, $C = R \oplus Ri$ is a plural domain, $H = C \oplus Cj = R \oplus Ri \oplus Rj \oplus Rk$ is a quaternion field.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

for any

$$x = x_0 + x_1 i + x_2 j + x_3 k \in H \tag{2}$$

can be uniquely represented as

$$x = x_0 + x_1 i + (x_2 + x_3 i) j = z_1 + z_2 j \tag{3}$$

among them $z_1, z_2 \in c$ similarly for $A \in M_n(H)$ can be unique-ly represented as $A = A_1 + A_2 j$,among them $A_1, A_2 \in M_n(H)$.

Definition 1: $A \in H^{m\times n}$,then A can be uniquely write

$$A = A_1 + A_2 j \ \left(A_1, A_2 \in C^{m\times n}\right) \tag{4}$$

we define A ‘s complex representation matrix to be

$$A_L = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \tag{5}$$

Definition 2: If $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in C^{2n\times 1}$,among them $\alpha_1, \alpha_2 \in C^{n\times 1}$,then α ‘s friend vector α^{υ} is defined as

$$\alpha^{\upsilon} = \begin{pmatrix} \overline{-\alpha_2} \\ \overline{\alpha_1} \end{pmatrix} \in C^{2n\times 1} \tag{6}$$

2. Quaternion Matrix Singular Value Decomposition

Definition 3: Assume $A \in H^{n\times n}$, if it exists

$$A^*A = AA^* = I_n \tag{7}$$

we call A quaternion unitary matrix.

Theorem: if $A \in H_r^{m\times n}$, there is a unitary matrix $U \in H^{m\times m}, V \in H^{n\times n}$ so that

$$A = U\sum V^* \tag{8}$$

Among them $\sum = diag(\sigma_1, \sigma_2, \cdots, \sigma_r, 0, 0, \cdots, 0) \in R^{m\times n}$, $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$ are r non-zero singular values of matrix A.

Proof: due to $(A^*A)_L = (A_L)^* A_L$ for the plural domain Hermite semi-definite matrices, by inference $(A_L)^* A_L$ the actual eigenvalues appear in pairs. Assume

$$\lambda_1 = \lambda_1' \ge \lambda_2 = \lambda'_2 \ge \cdots \ge \lambda_r = \lambda_r' > \lambda_{r+1} = \lambda_{r+1}' = \cdots = \lambda_n = \lambda_n' = 0 \tag{9}$$

Corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$ the standard orthogonal eigenvector is $x_1, x_2, \cdots, x_n$. $x_1^\upsilon, x_2^\upsilon, \cdots, x_n^\upsilon$ is standard orthogonal vector that corresponds to $\lambda_1', \lambda_2', \cdots, \lambda_n'$.

Defined by the inner product

$$\left\langle A_L x_i, A_L x_j \right\rangle = x_j^* (A_L)^* A_L x_i = \lambda x_j^* x_i = \begin{cases} \lambda_i, i = j \\ 0, i \ne j \end{cases} \tag{10}$$

$$\left\langle A_L x_i^\upsilon, A_L x_j^\upsilon \right\rangle = \left(x_j^\upsilon\right)^* (A_L)^* A_L x_i^\upsilon = \lambda_i' \left(x_j^\upsilon\right)^* x_i^\upsilon = \begin{cases} \lambda_i', i = j \\ 0, i \ne j \end{cases} \tag{11}$$

$$\left\langle A_L x_i, A_L x_j^\upsilon \right\rangle = 0 \tag{12}$$

So, we can see that

$$A_L x_1, A_L x_2, \cdots, A_L x_r, A_L x_1^\upsilon, A_L x_2^\upsilon, \cdots, A_L x_r^\upsilon \tag{13}$$

is non-zero and orthogonal to each other

Note

$$V_L = \left(x_1, x_2, \cdots, x_n, x_1^\upsilon, x_2^\upsilon, \cdots, x_n^\upsilon\right) \tag{14}$$

$$U_L = \left(\frac{A_L x_1}{\sqrt{\lambda_1}}, \frac{A_L x_2}{\sqrt{\lambda_2}}, \cdots, \frac{A_L x_r}{\sqrt{\lambda_1}}, y_{r+1}, \cdots, y_m, \frac{A_L x_1^\upsilon}{\sqrt{\lambda_1'}}, \frac{A_L x_2^\upsilon}{\sqrt{\lambda_2'}}, \cdots \frac{A_L x_r^\upsilon}{\sqrt{\lambda_r'}}, y_{r+1}^\upsilon, \cdots, y_m^\upsilon \right) \tag{15}$$

Among them $y_i, y_i^\upsilon (i = r+1, \cdots, m)$ is orthogonal vector set

$$\frac{A_L x_1}{\sqrt{\lambda_1}}, \frac{A_L x_2}{\sqrt{\lambda_2}}, \cdots, \frac{A_L x_r}{\sqrt{\lambda_1}} \; \frac{A_L x_1^\upsilon}{\sqrt{\lambda_1'}}, \frac{A_L x_2^\upsilon}{\sqrt{\lambda_2'}}, \frac{A_L x_3^\upsilon}{\sqrt{\lambda_3'}}, \cdots \frac{A_L x_r^\upsilon}{\sqrt{\lambda_r'}} \tag{16}$$

Expanded $C^{2m\times 1}$ standard orthogonal basis, obviously U_L, V_L both are unitary matrices, and there are

$$U_L^* A_L V_L = diag\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_r}, 0, \cdots, 0, \sqrt{\lambda_1'}, \sqrt{\lambda_2'} \cdots, \sqrt{\lambda_r'}, 0, \cdots, 0\right) = \sum_L \tag{17}$$

among them

$$\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r, 0, \cdots, 0), \sigma_i = \sqrt{\lambda_i}\,(i = 1,2,\cdots,r) \tag{18}$$

it is known from the proposition

$$A = U \Sigma V^* \tag{19}$$

among them $U \in H^{m\times m}, V \in H^{n\times n}$ is a unitary matrix, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ is r non-zero singular value of matrix A.

Definition 4 self-conjugated quaternion matrix A^*A versus AA^* the non-negative square root of the public eigenvalues is A 's singular value. Decomposition (8) is called A 's singular value decomposition.

Inference assume $A \in H_r^{m\times n}$, there are two standard orthogonal vector groups $u_s \in H^m$, $s = 1,2,\cdots,r$; $\upsilon_t \in H^n$ $t = 1,2,\cdots,r$ makes

$$A = \sigma_1 u_1 \upsilon_1^* + \sigma_2 u_2 \upsilon_2^* + \cdots + \sigma_r u_r \upsilon_r^* \tag{20}$$

among them $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ is r non-zero singular value of matrix A.

3. Static System Analysis of Singular Value Decomposition of Quaternion Matrix Literature References

The singular value decomposition (SVD) of static systems is considered in the case of electronic devices. It is assumed that there exists the following relationship between the voltage and current of an electronic device (i.e., the static system model)

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{F} \begin{pmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{21}$$

The Allowable v_1, v_2, i_1, i_2 Value of the Elements of a Matrix F.

If the voltage and current measurement devices used have the same accuracy (e.g. 1%), it is easy to detect the solutions of any set of measurement values that are or are not formulas (21) within the expected accuracy range. Assuming that another matrix expression is obtained by various methods:

$$\underbrace{\begin{pmatrix} 1 & -1 & 10^6 & 10^6 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{F} \begin{pmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

Obviously, only when the current is measured very accurately can a set v_1, v_2, i_1, i_2 of measurements be satisfied with the appropriate accuracy formula (22). For general cases of measurement 1% errors in current measurement, formula (22) is quite different from static system model (21): The voltage relationship given by formula (21) is $v_1 - v_2 = 0$, Because of the $i_1 + i_2 = 0.01$ measurement error, The voltage relationship given by formula (22) is $v_1 - v_2 + 10^4 = 0$. However, from an algebraic point of view, Formula (21) and Formula (22) are completely equivalent. Therefore, we hope to have some means to compare several algebraically equivalent models. The basic mathematical tool to solve this problem is singular value decomposition.

More generally, we consider the static system equation of a resistor:

$$F\begin{pmatrix} v \\ i \end{pmatrix} = 0 \tag{23}$$

In the formula, F is a matrix $m \times n$. In order to simplify the presentation, we have removed some invariable compensation items. Such an expression is very general and can be derived from some physical devices (e.g., linearized physical equations) and network equations. Singular value decomposition can be used to analyze the effect of matrix F on the exact and inaccurate parts of data. Let F the singular value be decomposed into

$$F = U^T \Sigma V \tag{24}$$

Thus, the components of the exact part and the inaccurate part are changed by the singular value $\sigma_1, \sigma_2, \cdots, \sigma_r, 0, \cdots, 0$ of the matrix F. If equation (23) is the exact specification of physical device design, then the F singular value decomposition of matrix will provide an algebraic equivalent, but it is the most reliable design equation in numerical value. Notice that U is an orthogonal matrix, so there are (3) and (4)

$$\Sigma V \begin{pmatrix} v \\ i \end{pmatrix} = O \tag{25}$$

If the diagonal matrix Σ is partitioned as

$$\Sigma = \begin{pmatrix} \Sigma_1 & O \\ O & O \end{pmatrix}$$

Orthogonal matrices V are partitioned accordingly at once $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ among $(A, \quad B)$ is V Top r row, Rule (25) can be written as $\begin{pmatrix} \Sigma_1 & O \\ O & O \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} v \\ i \end{pmatrix} = O$. Thus, we can obtain an algebraically equivalent expression to Formula (23), but numerically the most reliable expression:

$$(A, \quad B)\begin{pmatrix} v \\ i \end{pmatrix} = O \tag{26}$$

If Formula (26) is an imprecise type of physical device. Then there will be no zero-singular value on the diagonal line of the diagonal matrix.

At this time, we cannot use formula (26) directly. In this case, we need to modify the model by making all singular values $\sigma_s, \sigma_{s+1}, \cdots$ equal to zero. Among s is the smallest integer that satisfies σ_s / σ_1 the allowable accuracy (i.e. the measurement accuracy of the physical device) of elements less than the matrix F. Thus, the revised V top $s-1$ line in Formula (26) $(A, \quad B)$.

Relevant results show that such a modification can limit the variation of parameters to a preset error range.

Now consider the different expressions of a resistive multiterminal pair (resistance, conductance, mixing parameters, conduction and scattering, etc.) in order to find the best expression possible. For example, when end-to-end coordinates x and y time are used, the explicit representation of resistive multiterminal pairs is $y = Ax$, $\begin{pmatrix} y \\ x \end{pmatrix} = \Omega \begin{pmatrix} v \\ i \end{pmatrix}$.

The expressions of resistance, conductance, arbitrary mixed parameters or conduction can be obtained by choosing appropriate coordinate changes Ω. Thus, the condition number of the matrix A represents from x to y that the upper limit of the signal-to-noise ratio amplification factor. If A reversible, Then the condition number is also the upper limit of the signal-to-noise ratio amplification factor from y arrival x. Therefore, different expressions can be queued according to their condition number. This makes all parameterized expressions clear at a glance. Obviously, the optimal case is the conditional number $cond(A) = 1$ or A an orthogonal matrix (including a scale factor). A natural question is whether any multiterminal has an optimal expression for resistors, That is to say, whether there exists $cond(A) = 1$ such an orthogonal matrix A. To this end, let's look at the implicit expression of a resistor with n dimensional n end-pair: $F\begin{pmatrix} v \\ i \end{pmatrix} = 0$, $rank(F) = n$. Application of Singular Value Decomposition F Formula (24), Formula (26), in $r = n$. Choosing Orthogonal Coordinate Transform

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I/\sqrt{2} & I/\sqrt{2} \\ -I/\sqrt{2} & I/\sqrt{2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} \tag{27}$$

Thus, Available Ω orthogonality $\Omega^{-1} = Q^T$ Express the implicit expression (26) as

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} v \\ i \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} I/\sqrt{2} & -I/\sqrt{2} \\ I/\sqrt{2} & I/\sqrt{2} \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I & O \end{pmatrix} \begin{pmatrix} I/\sqrt{2} & -I/\sqrt{2} \\ I/\sqrt{2} & I/\sqrt{2} \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = O$$

That is to say $\begin{pmatrix} I/\sqrt{2} & -I/\sqrt{2} \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = 0 \Rightarrow y = x$ .also come to a conclusion, by using the singular value decomposition of (24), the formal transformation of formula (27) can be obtained. And through this orthogonal transformation, A numerical optimal display relationship can be obtained $y = x$.

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